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Some Methods for Solving the Fractional Differential Problem of Two Types of Fractional Functions

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Abstract: **In this paper, based on Jumarie's modified Riemann-Liouville (R-L) fractional derivative and a new multiplication of fractional analytic functions, we use some methods to find arbitrary order fractional derivative of two types of fractional functions. Moreover, our results are generalizations of classical calculus results.**

Keywords: **Jumarie's modified R-L fractional derivative, new multiplication, fractional analytic functions, fractional functions.**

I. INTRODUCTION

Fractional calculus is a branch of mathematical analysis, which studies several different possibilities of defining real order or complex order. In the second half of the 20th century, a large number of studies on fractional calculus were published in engineering literature. Fractional calculus is widely welcomed and concerned because of its applications in many fields such as mechanics, dynamics, control theory, physics, economics, viscoelasticity, electrical engineering, biology, and so on [1-11]

However, fractional calculus is different from ordinary calculus. The definition of fractional derivative is not unique. Common definitions include Riemann Liouville (R-L) fractional derivative, Caputo fractional derivative, Grunwald-Letnikov (G-L) fractional derivative and Jumarie's modification of R-L fractional derivative [12-16]. Because Jumarie type of R-L fractional derivative helps to avoid non-zero fractional derivative of constant function, it is easier to use this definition to connect fractional calculus with classical calculus.

In this paper, based on Jumarie's modified R-L fractional derivative and a new multiplication of fractional analytic functions, we use some methods to evaluate arbitrary order fractional derivative of the following two types of fractional functions:

$$arctanh_{\alpha}\left(\frac{2r}{1+r^2}cos_{\alpha}(x^{\alpha})\right),$$

and

$$arctan_{\alpha}\left(\frac{2r}{1-r^2}sin_{\alpha}(x^{\alpha})\right),$$

where $0<\alpha\leq 1$, r is a real number, and $|r|<1$. Moreover, our results are generalizations of ordinary calculus results.

II. PRELIMINARIES

At first, we introduce the fractional calculus used in this paper and its properties.

Definition 2.1 ([17])**:** Let $0<\alpha\leq 1$, and x_0 be a real number. The Jumarie type of Riemann-Liouville (R-L) α-fractional derivative is defined by

$$\left({}_{x_0}D_x^{\alpha}\right)[f(x)]=\frac{1}{\Gamma(1-\alpha)}\frac{d}{dx}\int_{x_0}^{x}\frac{f(t)-f(x_0)}{(x-t)^{\alpha}}dt\ , \quad (1)$$

where $\Gamma(\ )$ is the gamma function. On the other hand, for any positive integer m, we define $\left({}_{x_0}D_x^{\alpha}\right)^m[f(x)] = \left({}_{x_0}D_x^{\alpha}\right)\left({}_{x_0}D_x^{\alpha}\right)\cdots\left({}_{x_0}D_x^{\alpha}\right)[f(x)]$, the m-th order α-fractional derivative of $f(x)$.

Proposition 2.2 ([18])**:** *If α, β, x_0, C are real numbers and $\beta \geq \alpha > 0$, then*

$$\left({}_{x_0}D_x^{\alpha}\right)\left[(x-x_0)^{\beta}\right] = \frac{\Gamma(\beta+1)}{\Gamma(\beta-\alpha+1)}(x-x_0)^{\beta-\alpha}, \tag{2}$$

and

$$\left({}_{x_0}D_x^{\alpha}\right)[C] = 0. \tag{3}$$

Definition 2.3 ([19])**:** If x, x_0, and a_n are real numbers for all n, $x_0 \in (a,b)$, and $0 < \alpha \leq 1$. If the function $f_\alpha: [a,b] \to R$ can be expressed as an α-fractional power series, that is, $f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}$ on some open interval containing x_0, then we say that $f_\alpha(x^\alpha)$ is α-fractional analytic at x_0. Furthermore, if $f_\alpha: [a,b] \to R$ is continuous on closed interval $[a,b]$ and it is α-fractional analytic at every point in open interval (a,b), then f_α is called an α-fractional analytic function on $[a,b]$.

In the following, we introduce a new multiplication of fractional analytic functions.

Definition 2.4 ([20])**:** If $0 < \alpha \leq 1$. Assume that $f_\alpha(x^\alpha)$ and $g_\alpha(x^\alpha)$ are two α-fractional power series at $x = x_0$,

$$f_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}, \tag{4}$$

$$g_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha}. \tag{5}$$

Then

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{\Gamma(n\alpha+1)}(x-x_0)^{n\alpha} \\ &= \sum_{n=0}^{\infty} \frac{1}{\Gamma(n\alpha+1)}\left(\sum_{m=0}^{n}\binom{n}{m} a_{n-m}b_m\right)(x-x_0)^{n\alpha}. \end{aligned} \tag{6}$$

Equivalently,

$$\begin{aligned} & f_\alpha(x^\alpha) \otimes_\alpha g_\alpha(x^\alpha) \\ &= \sum_{n=0}^{\infty} \frac{a_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \otimes_\alpha \sum_{n=0}^{\infty} \frac{b_n}{n!}\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n} \\ &= \sum_{n=0}^{\infty} \frac{1}{n!}\left(\sum_{m=0}^{n}\binom{n}{m} a_{n-m}b_m\right)\left(\frac{1}{\Gamma(\alpha+1)}(x-x_0)^{\alpha}\right)^{\otimes_\alpha n}. \end{aligned} \tag{7}$$

Definition 2.5 ([21])**:** If $0 < \alpha \leq 1$, then the α-fractional exponential function is defined by

$$E_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{x^{n\alpha}}{\Gamma(n\alpha+1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha n}. \tag{8}$$

And the α-fractional cosine function and α-fractional sine function are defined as follows:

$$cos_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n\alpha}}{\Gamma(2n\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha 2n}, \tag{9}$$

and

$$sin_\alpha(x^\alpha) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{(2n+1)\alpha}}{\Gamma((2n+1)\alpha+1)} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}\left(\frac{1}{\Gamma(\alpha+1)}x^{\alpha}\right)^{\otimes_\alpha (2n+1)}. \tag{10}$$

Theorem 2.6 (fractional Euler's formula)([22])**:** *If $0 < \alpha \leq 1$, and $i = \sqrt{-1}$, then*

$$E_\alpha(ix^\alpha) = cos_\alpha(x^\alpha) + i sin_\alpha(x^\alpha). \tag{11}$$

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Theorem 2.7 (fractional DeMoivre's formula)([23])**:** *If* $0 < \alpha \leq 1$, *and* p *is an integer, then*

$$[cos_\alpha(x^\alpha) + isin_\alpha(x^\alpha)]^{\otimes_\alpha p} = cos_\alpha(px^\alpha) + isin_\alpha(px^\alpha). \quad (12)$$

Notation 2.8: If the complex number $z = p + iq$, where p, q are real numbers. p is the real part of z, and denoted by $\text{Re}(z)$; q is the imaginary part of z, and denoted by $\text{Im}(z)$.

Definition 2.9 ([24])**:** The smallest positive real number T_α such that $E_\alpha(iT_\alpha) = 1$, is called the period of $E_\alpha(ix^\alpha)$.

III. MAIN RESULTS

In this section, based on Jumarie type of R-L fractional derivative and a new multiplication of fractional analytic functions, we use some techniques to find arbitrary order fractional derivative of two types of fractional functions. At first, we need two lemmas.

Lemma 3.1: *If* $0 < \alpha \leq 1$, *r is a real number, and* $|r| < 1$, *then*

$$arctanh_\alpha(rE_\alpha(ix^\alpha)) = \frac{1}{2}arctanh_\alpha\left(\frac{2r}{1+r^2}cos_\alpha(x^\alpha)\right) + i \cdot \frac{1}{2}arctan_\alpha\left(\frac{2r}{1-r^2}sin_\alpha(x^\alpha)\right). \quad (13)$$

Proof $arctanh_\alpha(rE_\alpha(ix^\alpha))$

$$= \frac{1}{2}Ln_\alpha\left([1 + rE_\alpha(ix^\alpha)] \otimes_\alpha [1 - rE_\alpha(ix^\alpha)]^{\otimes_\alpha(-1)}\right)$$

$$= \frac{1}{2}Ln_\alpha\left([1 + rcos_\alpha(x^\alpha) + irsin_\alpha(x^\alpha)] \otimes_\alpha [1 - rcos_\alpha(x^\alpha) - irsin_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right)$$

$$= \frac{1}{2}Ln_\alpha\left(\begin{matrix}[1 + rcos_\alpha(x^\alpha) + irsin_\alpha(x^\alpha)] \otimes_\alpha [1 - rcos_\alpha(x^\alpha) + irsin_\alpha(x^\alpha)] \\ \otimes_\alpha [1 + r^2 - 2rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\end{matrix}\right)$$

$$= \frac{1}{2}Ln_\alpha\left([(1 - r^2) + i(2rsin_\alpha(x^\alpha))] \otimes_\alpha [1 + r^2 - 2rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right)$$

$$= \frac{1}{2}Ln_\alpha\left(\begin{matrix}\left[(1 - r^2)^2 + 4r^2(sin_\alpha(x^\alpha))^{\otimes_\alpha 2}\right]^{\otimes_\alpha\left(\frac{1}{2}\right)} \otimes_\alpha [1 + r^2 - 2rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)} \\ \otimes_\alpha \left\{[(1 - r^2) + i(2rsin_\alpha(x^\alpha))] \otimes_\alpha \left[(1 - r^2)^2 + 4r^2(sin_\alpha(x^\alpha))^{\otimes_\alpha 2}\right]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right\}\end{matrix}\right)$$

$$= \frac{1}{4}Ln_\alpha\left([1 + r^2 + 2rcos_\alpha(x^\alpha)] \otimes_\alpha [1 + r^2 - 2rcos_\alpha(x^\alpha)]^{\otimes_\alpha(-1)}\right)$$

$$+ \frac{1}{2}Ln_\alpha\left([(1 - r^2) + i(2rsin_\alpha(x^\alpha))] \otimes_\alpha \left[(1 - r^2)^2 + 4r^2(sin_\alpha(x^\alpha))^{\otimes_\alpha 2}\right]^{\otimes_\alpha\left(-\frac{1}{2}\right)}\right)$$

$$= \frac{1}{2}arctanh_\alpha\left(\frac{2r}{1+r^2}cos_\alpha(x^\alpha)\right) + i \cdot \frac{1}{2}arctan_\alpha\left(\frac{2r}{1-r^2}sin_\alpha(x^\alpha)\right).$$

q.e.d.

Lemma 3.2: *If* $0 < \alpha \leq 1$, *r is a real number, and* $|r| < 1$, *then*

$$arctanh_\alpha\left(\frac{2r}{1+r^2}cos_\alpha(x^\alpha)\right) = 2\sum_{n=0}^{\infty}\frac{1}{2n+1}r^{2n+1}cos_\alpha((2n+1)x^\alpha). \quad (14)$$

And

$$arctan_\alpha\left(\frac{2r}{1-r^2}sin_\alpha(x^\alpha)\right) = 2\sum_{n=0}^{\infty}\frac{1}{2n+1}r^{2n+1}sin_\alpha((2n+1)x^\alpha). \quad (15)$$

Proof $arctanh_\alpha\left(\frac{2r}{1+r^2}cos_\alpha(x^\alpha)\right)$

$$= 2\text{Re}[arctanh_\alpha(rE_\alpha(ix^\alpha))]$$

$$= 2\mathrm{Re}\left[\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(rE_\alpha(ix^\alpha)\right)^{\otimes_\alpha(2n+1)}\right]$$

$$= 2\mathrm{Re}\left[\sum_{n=0}^{\infty}\frac{1}{2n+1}r^{2n+1}E_\alpha(i(2n+1)x^\alpha)\right]$$

$$= 2\sum_{n=0}^{\infty}\frac{1}{2n+1}r^{2n+1}cos_\alpha((2n+1)x^\alpha)\ .$$

And

$$arctan_\alpha\left(\frac{2r}{1-r^2}sin_\alpha(x^\alpha)\right)$$

$$= 2\mathrm{Im}\left[arctanh_\alpha\left(rE_\alpha(ix^\alpha)\right)\right]$$

$$= 2\mathrm{Im}\left[\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(rE_\alpha(ix^\alpha)\right)^{\otimes_\alpha(2n+1)}\right]$$

$$= 2\mathrm{Im}\left[\sum_{n=0}^{\infty}\frac{1}{2n+1}r^{2n+1}E_\alpha(i(2n+1)x^\alpha)\right]$$

$$= 2\sum_{n=0}^{\infty}\frac{1}{2n+1}r^{2n+1}sin_\alpha((2n+1)x^\alpha)\ .$$ q.e.d.

Theorem 3.3: *If* $0<\alpha\leq 1$, r *is a real number,* $|r|<1$, *and* m *is any positive integer, then*

$$\left({}_0D_x^\alpha\right)^m\left[arctanh_\alpha\left(\frac{2r}{1+r^2}cos_\alpha(x^\alpha)\right)\right] = 2\sum_{n=0}^{\infty}(2n+1)^{m-1}r^{2n+1}cos_\alpha\left((2n+1)x^\alpha + m\cdot\frac{T_\alpha}{4}\right). \tag{16}$$

And

$$\left({}_0D_x^\alpha\right)^m\left[arctan_\alpha\left(\frac{2r}{1-r^2}sin_\alpha(x^\alpha)\right)\right] = 2\sum_{n=0}^{\infty}(2n+1)^{m-1}r^{2n+1}sin_\alpha\left((2n+1)x^\alpha + m\cdot\frac{T_\alpha}{4}\right). \tag{17}$$

Proof

$$\left({}_0D_x^\alpha\right)^m\left[arctanh_\alpha\left(\frac{2r}{1+r^2}cos_\alpha(x^\alpha)\right)\right]$$

$$= \left({}_0D_x^\alpha\right)^m\left[2\sum_{n=0}^{\infty}\frac{1}{2n+1}r^{2n+1}cos_\alpha((2n+1)x^\alpha)\right]$$

$$= 2\sum_{n=0}^{\infty}(2n+1)^{m-1}r^{2n+1}cos_\alpha\left((2n+1)x^\alpha + m\cdot\frac{T_\alpha}{4}\right).$$

And

$$\left({}_0D_x^\alpha\right)^m\left[arctan_\alpha\left(\frac{2r}{1-r^2}sin_\alpha(x^\alpha)\right)\right]$$

$$= \left({}_0D_x^\alpha\right)^m\left[2\sum_{n=0}^{\infty}\frac{1}{2n+1}r^{2n+1}sin_\alpha((2n+1)x^\alpha)\right]$$

$$= 2\sum_{n=0}^{\infty}(2n+1)^{m-1}r^{2n+1}sin_\alpha\left((2n+1)x^\alpha + m\cdot\frac{T_\alpha}{4}\right).$$ q.e.d.

IV. CONCLUSION

In this paper, based on Jumarie type of R-L fractional derivative and a new multiplication of fractional analytic functions, we use some methods to obtain arbitrary order fractional derivative of two types of fractional functions. In fact, our results are generalizations of ordinary calculus results. In the future, we will continue to use our methods to solve problems in engineering mathematics and fractional differential equations.

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